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On the positive Pell equation y^2=12x^2+13

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Abstract
The binary quadratic equation represented by the positive pellian y^2=12x^2+13 is analyzed for its distinct integer solutions. A few interesting among the solutions are given. Further, employing the solution of the above hyperbola, parabola and special Pythagorean triangle.

Keywords: binary quadratic, hyperpola, parabola, integralssolutions, pell equation

Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non_trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems. One many refer [5-20]. In this communication. Yet another interesting hyperpola gives by $y^2 = 12x^2 + 13$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of analysis

The positive bell equation representing hyperbola under consideration is,

$$y^2 = 12x^2 + 13 \rightarrow \qquad (1)$$

The smallest positive integer solution is $x_0 = 1, y_0 = 5$

The general solution (x_n, y_n) of (1) is given by

$$y_n = \frac{1}{2} f_n$$

$$x_n = \frac{1}{2\sqrt{12}} g_n$$

Where,

$$f_n = [(7+2\sqrt{12})^{n+1} + (7-2\sqrt{12})^{n+1}]$$

$$g_n = [(7+2\sqrt{12})^{n+1} - (7-2\sqrt{12})^{n+1}]$$

The recurrence relations satisfied by the solutions are given by

$$x_{n+1} - 14x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 14y_{n+2} + y_{n+3} = 0$$

Some numerical examples of x&y satisfying (1) are given in the table below.

Table 1

S.No	x_n	y_n
0	1	5
1	17	59
2	237	821
3	4012	11477
4	55931	194288

For the above table, we observe some interesting relations among the solutions which are presented below.

1. The x_n values are alternatively odd and even
2. The y_n valves are always odd

3. Each of the following expressions is a nasty number.

$*[30x_{2n+3} - 354x_{2n+2} + 156]/13$

$*[30x_{2n+4} - 4926x_{2n+2} + 2184]/182$

$*[60y_{2n+3} - 2448x_{2n+2} + 1092]/91$

$*[60y_{2n+4} - 34128x_{2n+2} + 15132]/1261$

$*[708x_{2n+4} - 9852x_{2n+3} + 312/26$

$*[708y_{2n+2} - 144x_{2n+3} + 1092]/91$

$*[708y_{2n+3} - 2448x_{2n+3} - 156]/13$

$*[708y_{2n+4} - 34128x_{2n+3} + 1092]/91$

$*[9852y_{2n+2} - 144x_{2n+3} + 15132/1261$

$*[9852y_{2n+2} - 2448x_{2n+4} + 1092]/91$

$*[9852y_{2n+3} - 34128x_{2n+4} + 156]/91$

$*[2448y_{2n+2} - 144y_{2n+3} + 3744]/312$

$*[34128y_{2n+2} - 144y_{2n+4} + 52416]/4368$

$*[34128y_{2n+3} - 2448y_{2n+4} + 3744]/312$

4. Each of the following expressions is a cubical integer

$*169[5x_{3n+4} - 59x_{3n+3} + 15x_{n+2} - 177x_{n+1}]$

$*16562[5y_{3n+4} - 204x_{3n+3} + 15y_{n+2} - 612x_{3n+3}]$

$*3180242[5y_{3n+5} - 2844x_{3n+3} + 15y_{n+3} - 8532x_{n+1}]$

$*1352[59x_{3n+5} - 821x_{3n+4} + 177x_{n+3} - 2463x_{n+2}]$

$*16562[59y_{3n+3} - 12x_{3n+4} + 177y_{n+1} - 36x_{n+2}]$

$*338[59y_{3n+4} - 204x_{3n+5} + 246y_{n+2} - 612x_{n+2}]$

$*16562[59y_{3n+5} - 2844x_{3n+4} + 177y_{n+3} - 8532x_{n+2}]$

$*3180242[821y_{3n+3} - 12x_{3n+5} + 2463y_{n+1} - 36x_{n+3}]$

$*16562[821y_{3n+4} - 204x_{3n+5} + 2463y_{n+2} - 612x_{n+3}]$

$*338[1642y_{3n+5} - 2844x_{3n+5} + 2463y_{n+3} - 8532x_{n+3}]$

$*194688[204y_{3n+3} - 12y_{3n+4} + 36y_{n+2} + 612y_{n+1}]$

$*457906176[237y_{3n+3} - y_{3n+5} + 711y_{n+1} - 3y_{n+3}]$

$*194688[2844y_{3n+4} - 204y_{3n+5} + 8532y_{n+2} - 612y_{n+3}]$

$*624[2844y_{3n+4} - 204y_{3n+5} + 8532y_{n+2} - 612y_{n+3}]$

5. Relation among the solutions

$*x_{n+1} = 14x_{n+2} - x_{n+3}$

$*7x_{n+1} = x_{n+2} - 2y_{n+1}$

$*x_{n+1} = 7x_{n+2} - 2y_{n+2}$

$* 7x_{n+1} = 97x_{n+2} - 2y_{n+3}$

$* 97x_{n+1} = x_{n+3} - 28y_{n+1}$

$* x_{n+1} = x_{n+3} - 4y_{n+2}$

$* x_{n+1} = 97x_{n+3} - 28y_{n+3}$

$* 24x_{n+1} = y_{n+2} - 7y_{n+1}$

$* 336x_{n+1} = y_{n+3} - 97y_{n+1}$

$* 24x_{n+1} = 7y_{n+3} - 97y_{n+2}$

$* 2x_{n+2} = 2y_{n+2} + x_{n+1}$

$* 97x_{n+2} = 7x_{n+3} - 2y_{n+1}$

$* 7x_{n+2} = x_{n+3} - 2y_{n+2}$

$* x_{n+2} = 7x_{n+3} - 2y_{n+3}$

$* 24x_{n+2} = 7y_{n+2} - y_{n+1}$

$* 48x_{n+2} = y_{n+2} - y_{n+1}$

$* 24x_{n+2} = y_{n+3} - 7y_{n+2}$

$* 7x_{n+3} = 28y_{n+2} + 7x_{n+1}$

$* 24x_{n+3} = 97y_{n+2} - 7y_{n+1}$

$* 336x_{n+3} = 97y_{n+3} - y_{n+1}$

$* 24x_{n+3} = 7y_{n+3} - y_{n+2}$

$* y_{n+1} = 14y_{n+2} - y_{n+3}$

$* 7y_{n+3} = 97x_{n+3} + 24x_{n+1}$

Remarkable observation

i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the table 1 below

Table 2

S.No.	(X,Y)	Hyperbola
1	$(17x_{n+1} - x_{n+2}, 5x_{n+2} - 59x_{n+1})$	$Y^2 - 12X^2 = 676$
2	$(474x_{n+1} - 2x_{n+3}, 5x_{n+3} - 821x_{n+1})$	$Y^2 - 12X^2 = 529984$
3	$(118x_{n+1} - 2y_{n+2}, 10y_{n+2} - 408x_{n+1})$	$Y^2 - 12X^2 = 33124$
4	$(1642x_{n+1} - 2y_{n+3}, 10y_{n+3} - 5688x_{n+1})$	$Y^2 - 12X^2 = 6360484$
5	$(474x_{n+3} - 34x_{n+3}, 118x_{n+3} - 1642x_{n+2})$	$Y^2 - 12X^2 = 2704$
6	$(10x_{n+2} - 34y_{n+1}, 118y_{n+1} - 24x_{n+2})$	$Y^2 - 12X^2 = 33124$
7	$(118x_{n+2} - 34y_{n+2}, 118y_{n+2} - 408x_{n+2})$	$Y^2 - 12X^2 = 676$
8	$1642x_{n+2} - 34y_{n+3}, 118y_{n+3} - 5688x_{n+2})$	$Y^2 - 12X^2 = 33124$

9	$(10x_{n+3}-474y_{n+1}, 1642y_{n+1}-24x_{n+3})$	$Y^2-12X^2=6360484$
10	$(118x_{n+3}-474y_{n+2}, 1642y_{n+2}-408x_{n+3})$	$Y^2-12X^2=33124$
11	$(1642x_{n+3}-474y_{n+3}, 1642y_{n+3}-5688x_{n+3})$	$Y^2-12X^2=676$
12	$(10y_{n+2}-118y_{n+1}, 408y_{n+1}-24y_{n+2})$	$Y^2-12X^2=389376$
13	$(10y_{n+3}-1642y_{n+1}, 5688y_{n+1}-24y_{n+3})$	$Y^2-12X^2=]76317696$
14	$(118y_{n+3}-1642y_{n+2}, 5688y_{n+2}-408y_{n+3})$	$Y^2-12X^2=389376$

ii) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table 2 below

Table 3

S.NO	(X,Y)	PARABOLA
1	$(17x_{n+1}-x_{n+2}, 5x_{2n+3}-59x_{2n+2})$	$13Y-12X^2=338$
2	$(474x_{n+1}-2x_{n+3}, 5x_{2n+4}-821x_{2n+2})$	$91Y-3X^2=66248$
3	$(118x_{n+1}-2y_{n+2}, 10y_{2n+3}-408x_{2n+2})$	$91Y-12X^2=1656$
4	$(1642x_{n+1}-2y_{n+3}, 10y_{2n+4}-5688x_{2n+2})$	$1261Y-12X^2=3180242$
5	$(474x_{n+2}-34x_{n+3}, 118x_{2n+4}-1642x_{2n+3})$	$13Y-6X^2=676$
6	$(10x_{n+2}-34y_{n+1}, 118y_{2n+2}-24x_{2n+3})$	$91Y-12X^2=16562$
7	$(118x_{n+2}-34y_{n+2}, 118y_{2n\|+3}-408x_{2n+3})$	$13Y-12X^2=3388$
8	$(1642x_{n+2}-34y_{n+3}, 118y_{2n+4}-5688x_{2n+3})$	$91Y-12X^2=16562$
9	$(10x_{n+3}-474y_{n+1}, 1642y_{2n+3}-24x_{2n+3})$	$1261Y-12X^2=3180242$
10	$(118x_{n+3}-474y_{n+2}, 1642y_{n+2}-408x_{2n+4})$	$91-12X^2=16562$
11	$(1642x_{n+2}-474y_{n+3}, 1642y_{2n+4}-5688x_{2n+4})$	$13Y-12X^2=338$
12	$(10y_{n+2}-118y_{n+1}, 408y_{2n+2}-24y_{2n+3})$	$26Y-X^2=16224$
13	$(10y_{n+3}-1642y_{n+1}, 5688y_{2n+2}-24y_{2n+4})$	$364Y-X^2=3179904$
14	$(118y_{n+3}-1642y_{n+2}, 5688y_{2n+3}-408y_{2n+4})$	$26Y-X^2=16224$

iii) Consider m= $x_{n+1}+y_{n+1}, n=x_{n+1}$ observe that m>n>0.Treat m, n as the genertors of the Pythagorean triangle $T(\alpha,\beta,\gamma)$

Where,

$\alpha=2pq$,

$\beta=p^2-q^2$,

$\gamma=p^2+q^2$

Then the following relations are observed

A. $\alpha-6\beta+5\gamma=-13$

B. $7\alpha-\gamma=\frac{24A}{p}-13$

C. $\frac{2A}{p}=x_{n+1}y_{n+1}$

D. $4\alpha-3\beta+2\gamma=\frac{12A}{p}-13$

Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation is $y^2 = 12x^2 + 13$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of positive pell equation and determine their integer solutions along with suitable properties.

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