



## Lebesgue decomposition theorem and its extension to signed measures

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### Abstract

By Lebesgue Decomposition Theorem for any  $\sigma$ -finite measurable space  $(X, \mathcal{A}, \mu)$  we prove that if  $\nu$  be any other  $\sigma$ -finite measure on  $\mathcal{A}$  then there exist a unique pair of measures  $(\nu_0, \nu_1)$  such that  $\nu = \nu_0 + \nu_1$  and  $\nu_0 \perp \mu$  and  $\nu_1 \ll \mu$ . The result can be extended to signed measures also.

**Keywords:** signed measure, measurable space, hahn-decomposition, orthogonal measures

### Introduction

**Definition:** Let  $\mathcal{A}$  be any  $\sigma$ -algebra on the set  $X$ .  $\vartheta$  be a set function defined on  $\mathcal{A}$  s.t.

- (1)  $-\infty \leq \vartheta \leq \infty$  and  $\vartheta$  takes at most one of the values  $-\infty$  and  $\infty$ .
- (2)  $\vartheta(\phi) = 0$
- (3)  $\vartheta$  is Countably additive, then  $\vartheta$  is called a Signed Measure on the space  $(X, \mathcal{A}, \vartheta)$ .

**Example 1:** Let  $\mu$  be any measure and  $\vartheta = -\mu$ , then  $\vartheta$  is a signed measure.

2. Let  $\lambda$  and  $\mu$  be any two measures s.t. at least one of them is finite. Take  $\vartheta = \lambda - \mu$ . Then  $\vartheta$  is a signed measure.

**Properties:** (1)  $\vartheta$  is finitely additive.

**Proof:** Let  $E_1, E_2, \dots, E_n$  be finitely many disjoint measurable sets. And  $E = \bigcup_{i=1}^n E_i$ , Define  $E_i = \phi$  for  $i > n$ . Then  $(E_i)$  is a disjoint sequence of measurable sets, Hence  $\vartheta(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \vartheta(E_i)$ .

(2) If  $\vartheta(E)$  is finite and  $F$  is a measurable subset of  $E$ , then  $\vartheta(F)$  is finite.

**Proof:**  $E = F \cup (E-F) \Rightarrow \vartheta(E) = \vartheta(F) + \vartheta(E-F)$ , Since  $\vartheta(E)$  is finite it follows immediately that  $\vartheta(F)$  is finite.

(3)  $\vartheta$  is subtractive i.e. if  $A \supset B$  be the measurable sets, and  $\vartheta(B)$  is finite and then  $\vartheta(A-B) = \vartheta(A) - \vartheta(B)$ .

**Proof:**  $A = B \cup (A-B) \Rightarrow \vartheta(A) = \vartheta(B) + \vartheta(A-B) \Rightarrow \vartheta(A-B) = \vartheta(A) - \vartheta(B)$ .

**Definition:** Let  $N$  be any measurable set such that  $\vartheta(M) = 0$  for every measurable subset  $M$  of  $N$ . Then  $N$  is called a null set of  $\vartheta$ .

### Remark

1. A measurable sub set of null set is a null set.
2. A countable union of null sets is a null set.

Let  $\{N_k\}$  be any sequence of null sets for  $\vartheta$ , write  $N = \bigcup_{k=1}^{\infty} N_k$  and  $M$  be any measurable subset of  $N$ . Define  $M_1 = N_1, M_2 = N_2 - N_1, M_3 = N_3 - (N_1 \cup N_2)$  .....

Then  $\{M_k\}$  is a disjoint sequence of Null sets, and  $\bigcup_{k=1}^{\infty} M_k = \bigcup_{k=1}^{\infty} N_k = N$   
 $M \subset N \Rightarrow M = M \cap N = M \cap [\bigcup_{k=1}^{\infty} M_k] = \bigcup_{k=1}^{\infty} [M \cap M_k]$

$$\Rightarrow \vartheta(M) = \sum_{k=1}^{\infty} \vartheta([M \cap M_k]) = 0. \quad [\text{As } M_k \text{ is a null set}]$$

Proved.

**Lemma:** Suppose  $\nu \perp \mu$  and  $\nu \ll \mu$  then  $\nu = 0$ .

**Proof:** Since  $\nu \perp \mu$ , we can find disjoint measurable sets A and B such that  $X = A \cup B$  and  $\nu = 0$  on A and  $\mu = 0$  on B.

Let E be any measurable sub set of X, then  $E = E \cap X = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$

Then  $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap B)$  [Because  $\nu(E \cap A) = 0$ ]

As  $\mu(E \cap B) = 0$  and  $\nu \ll \mu$  we have  $\nu(E \cap B) = 0 \Rightarrow \nu(E) = 0 \forall$  measurable sub set E of X.  $\Rightarrow \nu = 0$ .

**Note:** Above said theorem can also be stated as that Hahn-Decomposition is unique up to null sets.

**Definition:** Let  $\nu$  be any signed measure and S be any measurable subset of X. if X-S is a null set for  $\nu$ . Then we say that  $\nu$  is supported on S.

**Definition:** Let  $\nu_1, \nu_2$  be two signed measures, then there exist a measurable set S s.t.  $\nu_1$  is supported on S and  $\nu_2$  is supported on X-S. Then we say that  $\nu_1$  orthogonal or singular to  $\nu_2$  and we write as  $\nu_1 \perp \nu_2$ .

Remark:  $\nu_1 \perp \nu_2 \Leftrightarrow \nu_2 \perp \nu_1$  therefore we can say that  $\nu_1$  and  $\nu_2$  are mutually orthogonal.

**Lebesgue Decomposition Theorem:** Let  $(X, \mathcal{A}, \mu)$  be any  $\sigma$ -finite measure space. Let  $\nu$  be any other  $\sigma$ -finite measure on  $\mathcal{A}$  then there exist a unique pair of measures  $(\nu_0, \nu_1)$  such that  $\nu = \nu_0 + \nu_1$  and  $\nu_0 \perp \mu$  and  $\nu_1 \ll \mu$ .

**Proof:** Define  $\lambda = \mu_+$ , then  $\lambda$  is a  $\sigma$ -finite measure and it is obvious that  $\mu \ll \lambda$  and  $\nu \ll \lambda$ .

Hence by Radon Nikodym Theorem we can find a non-negative measurable functions  $f$  and  $g$  such that  $\mu(E) = \int_E f d\lambda$  and  $\nu(E) = \int_E g d\lambda$  for every measurable set E of X.

Define  $A = \{f = 0\}$  and  $B = \{f > 0\}$  then A and B are disjoint and  $X = A \cup B$ .

Since  $f = 0$  on A we get  $\int_A f d\lambda = 0 \Rightarrow \mu(A) = 0$ .

Define  $\nu_0(E) = \nu(E \cap A)$  then  $\nu_0$  is a measure and  $\nu_0(B) = \nu(B \cap A) = 0 \Rightarrow \nu_0(B) = 0$

$$[B \cap A = \phi]$$

Shows that  $\nu_0 \perp \mu$  ..... (1)

Define  $\nu_1(E) = \nu(E \cap B)$  then  $\nu_1$  is a measure and  $\nu(E) = \nu(E \cap X) = \nu(E \cap (A \cup B)) = \nu(E \cap A) + \nu(E \cap B) = \nu_0(E) + \nu_1(E)$  for all measurable subsets E of X

Shows that  $\nu = \nu_0 + \nu_1$  ..... (2)

To show  $\nu_1 \ll \mu$

Suppose  $\mu(E) = 0 \Rightarrow \int_E f d\lambda = 0 \Rightarrow f = 0$  a.e.  $[\lambda]$  on E, Since  $f > 0$  on  $E \cap B$

$\Rightarrow \lambda(E \cap B) = 0$  [Because  $E \cap B$  is a null set]

$\Rightarrow \nu(E \cap B) = 0$  as  $\nu \ll \lambda \Rightarrow \nu_1(E) = 0$  shows that  $\nu_1 \ll \mu$  ..... (3)

**Uniqueness:** Suppose we also have  $\nu = \nu'_0 + \nu'_1$  where  $\nu'_0$  and  $\nu'_1$  are measures on  $\mathcal{A}$  and  $\nu'_0 \perp \mu, \nu'_1 \ll \mu$ . As  $\mu$  and  $\nu$

both are  $\sigma$ -finite there exist a disjoint sequence  $\{X_n\}$  of measurable sets s.t.  $X = \bigcup_{n=1}^{\infty} X_n$   
 And  $\mu(X_n) < \infty, \nu(X_n) < \infty, \forall n$ .

This implies that  $\nu_0, \nu_1, \nu'_0, \nu'_1$  are all finite on  $X_n$ .

And on  $X_n$  we have  $\nu = \nu_0 + \nu_1, \nu = \nu'_0 + \nu'_1$

$\Rightarrow \nu_0 + \nu_1 = \nu'_0 + \nu'_1 \Rightarrow \nu_0 - \nu'_0 = \nu'_1 - \nu_1$  on  $X_n$

$\nu_0 \perp \mu, \nu'_0 \perp \mu \Rightarrow \nu_0 - \nu'_0 \perp \mu$ , on the other hand  $\nu_1 \ll \mu$  and  $\nu'_1 \ll \mu$

Therefore  $\nu'_1 - \nu_1 \ll \mu \Rightarrow \nu_0 - \nu'_0 \ll \mu$ . By the above lemma we obtain  $\nu_0 - \nu'_0 = 0$

$\Rightarrow \nu_0 = \nu'_0$  on  $X_n \Rightarrow \nu_0 = \nu'_0$  on  $X$  also.

Similarly  $\nu_1 = \nu'_1$  on  $X$  also. Hence Proved

**Lebesgue Decomposition Theorem for signed measures:** Let  $(X, \mathcal{A}, \mu)$  be any  $\sigma$ -finite measurable space, Let  $\nu$  be any finite signed measure, then there exist a unique pair  $(\nu_0, \nu_1)$  of finite signed measures s.t.  $\nu = \nu_0 + \nu_1, \nu_0 \perp \mu, \nu_1 \ll \mu$ .

**Proof:** Since  $\nu$  is finite it follows that  $\nu^+$  is a finite measure

Define  $\lambda = \mu + \nu^+$  then  $\lambda$  is a  $\sigma$ -finite measure and  $\mu \ll \lambda, \nu^+ \ll \lambda$ .

By Radon Nikodym Theorem we can find a non-negative measurable functions  $f$  and  $g$  such that  $\mu(E) = \int_E f d\lambda$  and  $\nu^+(E) = \int_E g d\lambda$  for every measurable set  $E$  of  $X$ .

Define  $A_1 = \{f = 0\}$  and  $B_1 = \{f > 0\}$  then  $A_1$  and  $B_1$  are disjoint measurable sets,

$X = A_1 \cup B_1$  as  $f = 0$  on  $A_1$  we have  $\int_{A_1} f d\lambda = 0 \Rightarrow \mu(A_1) = 0$

Let  $E$  be any measurable set such that  $\mu(E) = 0$  which gives  $\int_E f d\lambda = 0 \Rightarrow f = 0$  a.e.  $[\lambda]$  on  $E$ .

Since  $f > 0$  on  $E \cap B_1$ , we get  $\lambda(E \cap B_1) = 0 \Rightarrow \int_{E \cap B_1} g d\lambda = 0 \Rightarrow \nu^+(E \cap B_1) = 0$  [As  $\nu^+ \ll \lambda$ ]

Thus we obtain disjoint measurable sets  $A_1$  and  $B_1$  such that  $\mu(A_1) = 0, \nu^+(E \cap B_1) = 0$  whenever  $\mu(E) = 0, X = A_1 \cup B_1$ .....(1) In the same manner we can find measurable disjoint sets  $A_2$  and  $B_2$  such that  $X = A_2 \cup B_2, \mu(A_2) = 0, \nu^-(E \cap B_2) = 0$  whenever  $\mu(E) = 0$ .....(2)

Define  $A = A_1 \cup A_2$  and  $B = X - A$  then  $A$  and  $B$  are disjoint measurable sets and  $X = A \cup B$

Define  $\nu_0(E) = \nu(E \cap A), \nu_1(E) = \nu(E \cap B)$  for any measurable set  $E$ .

As  $\nu$  is a finite signed measure it is clear that  $\nu_0, \nu_1$  both are signed measures for any measurable sub set  $E$  of  $X$

$E = E \cap X = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$

$\Rightarrow \nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu_0(E) + \nu_1(E) \Rightarrow \nu = \nu_0 + \nu_1$

From  $A = A_1 \cup A_2$  we get  $\mu(A) \leq \mu(A_1) + \mu(A_2) \Rightarrow \mu(A) = 0$

Let  $E$  be any measurable sub set of  $B$  then  $\nu_0(E) = \nu(E \cap A) = \nu(\phi) = 0$

Shows that  $\nu_0 \perp \mu$

To show that  $\nu_1 \ll \mu$ , assume that  $\mu(E) = 0$ , then from (1) and (2), we get

$\nu^+(E \cap B_1) = 0, \nu^-(E \cap B_2) = 0$

$E \cap B = E \cap (A^c) = E \cap [(A_1 \cup A_2)^c] = E \cap [(A_1)^c \cap (A_2)^c] = E \cap [B_1 \cap B_2]$

$= (E \cap B_1) \cap (E \cap B_2)$  [From (1) and (2)]

$\Rightarrow E \cap B \subset E \cap B_1, E \cap B \subset E \cap B_2$

$\Rightarrow \nu^+(E \cap B) = 0, \nu^-(E \cap B) = 0$  [From (1) and (2)]

$\Rightarrow \nu(E \cap B) = 0 \Rightarrow \nu_1(E) = 0$

Shows that  $\nu_1 \ll \mu$ .

**Uniqueness:** Suppose we also have  $\nu = \nu'_0 + \nu'_1$  where  $\nu'_0$  and  $\nu'_1$  are finite signed measures on  $\mathcal{A}$  and  $\nu'_0 \perp \mu, \nu'_1 \ll \mu$ .

As  $\mu$  and  $\nu$  both are  $\sigma$ -finite there exist a disjoint sequence  $\{X_n\}$  of measurable sets s.t.  $X = \bigcup_1^\infty X_n$  and  $\mu(X_n) < \infty, \nu(X_n) < \infty, \forall n$ .

This implies  $\nu_0, \nu_1, \nu'_0, \nu'_1$  are all finite on  $X_n$ . And on  $X_n$  we have  $\nu = \nu_0 + \nu_1, \nu = \nu'_0 + \nu'_1$

$\Rightarrow \nu_0 + \nu_1 = \nu'_0 + \nu'_1 \Rightarrow \nu_0 - \nu'_0 = \nu'_1 - \nu_1$  on  $X_n$

$\nu_0 \perp \mu, \nu'_0 \perp \mu \Rightarrow \nu_0 - \nu'_0 \perp \mu$ , on the other hand  $\nu_1 \ll \mu$  and  $\nu'_1 \ll \mu$

therefore  $\nu'_1 - \nu_1 \ll \mu \Rightarrow \nu_0 - \nu'_0 \ll \mu$  by the above lemma we obtain  $\nu_0 - \nu'_0 = 0$

$\Rightarrow \nu_0 = \nu'_0$  on  $X_n \Rightarrow$  on  $\nu_0 = \nu'_0$  X also.

Similarly  $\nu_1 = \nu'_1$  on X. Hence Proved

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